



Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Review of Economic Dynamics 8 (2005) 178–194

Review of
Economic
Dynamics

www.elsevier.com/locate/red

Social security as Markov equilibrium in OLG models

Lorenzo Forni

Research Department, Bank of Italy, Via Nazionale 91, 00184 Rome, Italy

Received 26 December 2000; revised 1 September 2004

Available online 13 December 2004

Abstract

This paper studies the characteristics of intergenerational transfers in a standard overlapping generations model with short lived governments that care about the welfare of young generations only. A number of authors have shown that simple intergenerational games, in which in each period the current young generation plays as a dictator, are able to deliver political equilibria with social security even if the underlying competitive equilibrium is not dynamically inefficient. These authors have either derived pure steady state results or have relied on subgame-perfectness. This paper extends these results deriving Markov subgame perfect equilibria (i.e. that depend only upon the period t state variable, which is the stock of capital).

Non-Markov subgame perfect equilibria assume agents know all the past history of the game; they cannot predict when the social security system will emerge and whether or not it will eventually emerge; they prescribe that generations that never deviated may be punished. Markov equilibria, placing more restrictions on the structure of the game, are able to deliver solutions that do not suffer from these drawbacks. As the paper shows, however, Markov strategies may produce unstable dynamics.

© 2004 Elsevier Inc. All rights reserved.

JEL classification: C72; H55

Keywords: Social security; Overlapping generations models; Markov equilibria

E-mail address: lorenzo.forni@bancaditalia.it.

1094-2025/\$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.red.2004.10.003

1. Introduction

Intergenerational redistribution is a very important issue in the political debate: on the one hand pension systems can be manipulated for political purposes; on the other, it is not clear how a transfer scheme should be designed to be optimal and thus more secure from political pressures, considering that it should be flexible enough to adjust to exogenous (such as population or technological) shocks.

In this context, it is very important to understand the widespread role of intergenerational transfers and why pay-as-you-go (PAYGO) social security systems developed and became stable institutions of modern societies. In particular, the existence of PAYGO schemes seems puzzling, given that successive generations cannot subscribe any commitment to pay old age benefits to older generations. There are many explanations in the literature of why PAYGO social security have been introduced and then expanded. Some relate to the incompleteness of private annuity markets and adverse selection, others to the fact that PAYGO systems can help to overcome inefficient intergenerational credit markets. The classical solution to the puzzle is that, if the economy is on a dynamically inefficient path (such that the interest rate falls below the economy's growth rate), then the introduction of a PAYGO social security scheme is Pareto improving since it reduces the capital deepening. However, even in this case, a PAYGO social security scheme is a dynamically inconsistent agreement between successive generations. Young generations would be better off discontinuing the PAYGO scheme and setting up a new one. Hence the question arises of why PAYGO schemes survive.

Recently a growing literature has analyzed the role of PAYGO pension schemes in the context of majority voting in overlapping generations models. Azariadis and Galasso (1997), Galasso (1998), Cooley and Soares (1998) show that equilibria exist in which social security emerges through a majority voting mechanism. Boldrin and Rustichini (2000) show that "a PAYGO public pension plan is a subgame perfect equilibrium of a majority voting game in an OLG model with production and capital accumulation when the growth rate of total labor productivity and the initial stock of capital satisfy a certain set of restrictions."

These contributions show that a PAYGO system can be supported by a strategy which stipulates that a punishment, in the form of no old age transfer, must be inflicted to all generations that in the past have modified its rules. That is, a generation which votes to deviate from the prescribed level of social security payroll tax would be punished and would not receive any old age benefit. A generation which decides not to inflict the punishment would not receive any old age benefit as well (in this case, it would contribute to the system but would not receive any benefit back).

These equilibria have some limitations. In the first place, both steady states (Cooley and Soares, 1998) and dynamic (Boldrin and Rustichini, 2000) subgame perfect equilibria (SPE) assume agents know the complete past history of the game and use this information when deciding their actions. Secondly, they have not sharp implications: as Boldrin and Rustichini (2000) show, although the social security system can be sustained in equilibrium, the model cannot predict when it will emerge and whether or not it will eventually emerge. Third, they prescribe that the PAYGO is never started again if one generation de-

viated in the past; if having the PAYGO is beneficial to the generation that starts it and to the following ones, these would get punished by the strategy even if they never deviated.

In this paper I show that to support the social security system it is not necessary to condition the transfer to the elderly to their past behavior. The system can be supported by a simple negative relation between the level of saving of each generation and the amount of old age transfers that the same generation receive: if a generation decides to reduce the contribution to the system below the equilibrium level, it will increase its saving and will reach old age with a higher level of the capital stock (in the form, for example, of high private pension accounts); if the same generation would receive a reduced old age transfer as a result of the higher level of capital stock owned, it could be better off not reducing the contribution to the system in the first place. This simple rule in fact implements a punishment type strategy as in Boldrin and Rustichini (2000) that says: “check the capital stock; if it is too high, that means the current elderly did not transfer the amount of resources as they should have, hence lower the transfer to them.” The paper shows that this rule has a Markovian representation. Thus, the Markovian equilibrium requires simply to check the level of the capital stock and to punish agents in proportion of the amount of their deviation from the equilibrium path.

The model I present shares many of the characteristics of the one of Boldrin and Rustichini (2000): I assume as well that the median voter is a member of the young cohort, that young cohorts (or the governments representing the young) have to decide upon the payroll tax that the young have to pay to the old as a fraction of their wage, that this payroll tax can change with the capital stock and that saving and consumption are determined by individual maximization. The main difference is that I focus on Markov subgame perfect equilibrium (SPE). Focusing on Markov strategies I am able to restrict the set of potential equilibria, which in standard SPE are usually many. In fact, since the model entails a game among generations, the strategy chosen by the current government depends on the course of action it expects for the future (retirement benefits for the current young will in fact depend on the payroll tax that future young will decide to pay). In this set up, there may be many policy trajectories that are self-fulfilling. Markov equilibria, assuming that strategies depend only on the value of the state variable (which is the capital stock) put strong constraints on the set of admissible forecasts. Since the economy is stationary, it is reasonable to restrict the attention to policies that depend solely on the capital stock (and are thus independent of the period in which the government is in power), that is to policies of the type $\tau(k_t)$. In fact there is no reason why a government in power at time t_1 , when the capital stock is k_1 , should behave differently from a government that is in power at time t_2 , when the capital stock is k_2 , if $k_1 = k_2$.¹ Strategies that depend solely on the value of the prevailing state variable are usually referred to as Markov strategies.²

¹ These requirements are suggested also by Krusell et al. (1997), who argue that in a stationary environment the policy expected for period t ought to depend only on the value of the state variable expected in that period.

² A close reference to this paper is also Grossman and Helpman (1998). They present an overlapping generations model in which “successive governments are unable to pre-commit the future course of redistributive taxation,” and solve for the stationary equilibrium transfer policy. This paper differs in two ways. First, Grossman and Helpman assume that representatives seek to maximize a weighted sum of lifetime utilities of both workers and retirees. That is, they assign a positive weight to the old in the political process which I do not assume. In

Markovian equilibria in principle do not suffer from the mentioned limitations of more general subgame perfect ones. First, agents need to know only the current value of the state variable, which in the model is the stock of capital, and not all the past history of the game. Second, since the equilibrium strategy will set a relation between the payroll tax that young workers decide to pay and the stock of capital which is owned by the elderly, for each level of the capital stock there will be an equilibrium tax. Finally, the equilibrium implies that the level of the payroll tax changes continuously as the economy grows (or shrinks); if a generation deviates from the equilibrium path, this will affect the capital stock and the future course of payroll taxes, but it does not imply that the PAYGO will collapse forever.

I will show that the Markovian equilibrium adds on agents' reasons to support a PAYGO system. As in Boldrin and Rustichini (2000), the PAYGO can arise either due to the classical efficiency-enhancing role of PAYGO social security systems or to the fact that the transfer of resources from the young to the old reduces the level of capital; in this latter case, if the return per unit of capital has an elasticity greater than one, a reduction in saving increases the total return more than proportionally, thereby increasing lifetime income of the young generation (the "saving monopolist" argument). In the Markovian solution, a generation may want to increase the transfers to the elderly also as this would secure, through a reduction in its level of saving, an increase in old age benefits supplied by future generations. This is a continuous version of the punishment type strategy as in Boldrin and Rustichini (2000). It is different, since Boldrin and Rustichini (2000) assume agents either decide to pay the given payroll tax (set by the previous generation) or not to pay at all. It is worth noting that the punishment strategy needed to support the PAYGO as a political equilibrium actually leads to a higher steady state level of capital than in the model where payroll taxes are exogenously given. That is in equilibrium, since pension benefits are a decreasing function of the capital stock, a higher level of saving than in the competitive model is necessary in order to achieve consumption smoothing. Against all these pluses the paper shows that Markov strategies, when simple logarithmic utility and Cobb–Douglas production functions are assumed, may produce unstable dynamics.

The paper is organized as follows. Section 2 describes the model and Section 3 the equilibrium concept. Section 4 present the solution assuming a logarithmic utility function. In Section 5, I consider a more general utility function (CRRA) and I use numerical simulation techniques to approximate the policy function and to analyze the dynamic properties of the equilibrium. Section 6 concludes.

2. The model

I consider a standard two periods overlapping generations model. Agents work, consume, save and pay taxes in the first period. They are retired, consume, and receive old age benefits in the second one. To start with I assume a logarithmic utility function, and that

their model, the equilibrium is not stable if the weight on the old is set to zero. Second, Grossman and Helpman assume that the young do not consume and solve the model only for a linear production function, so that the interest rate and the wage rate are exogenous and fixed. On the contrary, I set up a more general model with a full-fledged characterization of the private sector and more general production and utility functions.

the production function is Cobb–Douglas, $Y = K^\alpha L^{1-\alpha}$; in per capita terms, $y = k^\alpha$ with $k = K/L$. It is assumed that capital depreciates completely in one period.

The economy is characterized by a PAYGO pension scheme that summarizes the issue of the intergenerational redistribution. Governments are assumed to be short-lived and to maximize lifetime utility of current workers (young). In any other respect the economy is competitive; in particular, individuals assume the payroll taxes as given when they decide upon their saving rate.

The representative young agent in period t solves:

$$\max_{c_{1t}, c_{2t}} u(c_{1t}, c_{2t}) = \lg[c_{1t}] + \beta \lg[c_{2t}] \quad (2.1)$$

subject to

$$\begin{aligned} c_{1t} &= w_t(1 - \tau_t) - S_t, \\ c_{2t} &= S_t(1 + r_{t+1}) + \hat{b}_{t+1}, \end{aligned}$$

where S_t is savings and $\hat{b}_{t+1} = b_{t+1}w_t$ is the old age benefit expressed as a replacement rate (b_{t+1}) of the first period wage. Given the Cobb–Douglas production function, the equilibrium conditions on the firm's side imply that: $w_t = (1 - \alpha)k_t^\alpha$ is the wage, $r_{t+1} = \alpha k_{t+1}^{\alpha-1} - 1$ is the interest rate (since it is assumed complete depreciation in one period). I assume also that the government cannot finance expenditures with debt, thus it has to meet the PAYGO budget constraint

$$\hat{b}_{t+1} = b_{t+1}w_t = \tau_{t+1}w_{t+1}(1 + n). \quad (2.2)$$

The tax rate τ_t is the policy instrument of the generational government. That is, the government at time t chooses the transfer to the current old, $\tau_t w_t$, in order to maximize the lifetime utility of his constituency (the young). This choice is conditional on the expected next period transfer, $\tau_{t+1}w_{t+1}$.³

Agents choose their saving and consumption functions by individual maximization taking τ_t and b_{t+1} as given, so that:⁴

³ The way I represent the intergenerational transfer, that is in term of the replacement rate b_{t+1} , is only for notational convenience, but is not restrictive. I could have represented the old age benefit as a function of the rate of return of the PAYGO system, that is such that

$$\hat{b}_{t+1} = \tau_t w_t(1 + i_{t+1}) \quad (2.3)$$

where i_{t+1} represents the second period expected return on first period contributions. Obviously

$$(1 + i_{t+1}) = \frac{\hat{b}_{t+1}}{\tau_t w_t} = \frac{\tau_{t+1} w_{t+1}(1 + n)}{\tau_t w_t} \quad (2.4)$$

and therefore

$$\hat{b}_{t+1} = \frac{\tau_{t+1} w_{t+1}(1 + n)}{\tau_t w_t} \tau_t w_t = \tau_{t+1} w_{t+1}(1 + n) \quad (2.5)$$

which is exactly (2.2).

⁴ The fact that saving is determined by competitive behavior rules out the possibility that the members of each generation collude in order to provide a smaller transfer without increasing their savings.

$$c_{1t} = \left(\frac{1}{1+\beta} \right) \left[(1 - \tau_t) + \frac{b_{t+1}}{(1+r_{t+1})} \right] w_t, \quad (2.6)$$

$$c_{2t} = \left(\frac{\beta}{1+\beta} \right) [(1 - \tau_t)(1 + r_{t+1}) + b_{t+1}] w_t, \quad (2.7)$$

$$S_t = \left(\frac{1}{1+\beta} \right) \left[\beta(1 - \tau_t) - \frac{b_{t+1}}{(1+r_{t+1})} \right] w_t. \quad (2.8)$$

The model is completed by the dynamic equation for the capital:

$$(1+n)k_{t+1} = S_t. \quad (2.9)$$

3. The equilibrium concept

As stated in the introduction, I consider an equilibrium in Markov strategies; that is, the policy function is of the type $\tau_t = \tau_t(k_t)$. I assume this function to be continuous and differentiable. Being the model stationary, it is reasonable to restrict our attention to those policies that depend solely on the capital stock, and also that do not depend on the period in which the government is in power, that is to policies of the type $\tau(k_t)$.

The policy chosen by the government at time t will depend on its expectation of the transfer policy of the government at time $t+1$. For example, assume that in equilibrium $\tau(k)$ (and the old age benefit \hat{b}) is a decreasing function of k (that is $\tau'(k) < 0$). Everything else equal, a higher expected old age transfer reduces the level of saving in the first period (see Eq. (2.8) for S_t) and therefore the capital that each generation is willing to pass to the following one. An equilibrium (at time t) entails a level of $\bar{\tau} = \tau(\bar{k}_{t+1})$ such that workers at time t are willing to save $S_t/(1+n) = \bar{k}_{t+1}$.

More generally, the problem can be stated in the following terms. As I show in the next section, agents' consumption in the first period, c_{1t} , can be expressed as a function of k_t , $\tau(k_t)$, $\tau(k_{t+1})$. Agents' consumption in the second period, c_{2t} , is going to depend on the same variables plus the prevailing interest rate in the second period, r_{t+1} . Since saving is defined as disposable income $w_t(1 - \tau_t)$ minus first period consumption, c_{1t} , saving is also a function of k_t , $\tau(k_t)$, $\tau(k_{t+1})$. Therefore, we can rewrite the problem as follows:

$$\tau(k_t) = T[\tilde{\tau}(k_{t+1})] \equiv \arg \max_{0 < \tau(\cdot) < 1} u(c_{1t}, c_{2t}) = \arg \max_{0 < \tau(\cdot) < 1} \Psi(k_t, \tau(k_t), \tilde{\tau}(k_{t+1})) \quad (3.1)$$

$$\text{such that } k_{t+1} = \Phi(k_t, \tau(k_t)) \quad (3.2)$$

where $\Psi(\cdot)$ is the indirect utility function written as a function solely of k_t , $\tau(k_t)$, and $\tilde{\tau}(k_{t+1})$. $\Phi(\cdot)$ is the (possibly) explicit form of the capital accumulation equation $(1+n) \times k_{t+1} = S_t(k_t, \tau(k_t), \tilde{\tau}(k_{t+1}))$.⁵ $\tilde{\tau}(k_{t+1})$ represents the time t government's expectation of the policy function of $t+1$. I therefore look for a policy function $\tau(k)$ that solves the

⁵ Note that it might not be possible in general to obtain an explicit form for the capital accumulation equation as in (3.2). The condition to have an explicit expression and therefore for the function $\Phi(\cdot)$ to exist is given by the Implicit Function Theorem, and it is $\frac{\partial}{\partial k_{t+1}} \{(1+n)k_{t+1} - S_t(k_t, \tau(k_t), \tau(k_{t+1}))\} \neq 0$ for all relevant values of the variables.

above fixed point problem, so that the expected next period policy, $\tilde{\tau}(\cdot)$, is equal to the best response to that policy, $\tau(\cdot)$.

4. Intergenerational transfers

As said, the choice of time t government will depend on its expectation of the transfer policy of the government at time $t + 1$. This expectation enters only in the old age replacement rate b_{t+1} . Thus, let us rewrite b_{t+1} in a more useful form. First, from the equilibrium conditions of the firm and the assumed Cobb–Douglas production function we have that

$$r_t = \alpha k_t^{\alpha-1} - 1 = \alpha \frac{y_t}{k_t} - 1, \quad (4.1)$$

$$w_t = (1 - \alpha)k_t^\alpha = (1 - \alpha)y_t = \left(\frac{1 - \alpha}{\alpha}\right)(1 + r_t)k_t. \quad (4.2)$$

Therefore, using (4.1), (4.2) and the capital accumulation expression (2.9), we have

$$\begin{aligned} b_{t+1} &= \frac{\hat{b}_{t+1}}{w_t} = \frac{\tau(k_{t+1})w_{t+1}(1+n)}{w_t} \\ &= \frac{\tau(k_{t+1})\left(\frac{1-\alpha}{\alpha}\right)(1+r_{t+1})(1+n)k_{t+1}}{w_t} \\ &= \tau(k_{t+1})\left(\frac{1-\alpha}{\alpha}\right)(1+r_{t+1})\left(\frac{1}{1+\beta}\right)\left[\beta(1-\tau(k_t)) - \frac{b_{t+1}}{(1+r_{t+1})}\right]. \end{aligned} \quad (4.3)$$

Thus, solving for b_{t+1} ,

$$b_{t+1} = \beta(1 - \tau(k_t))(1 + r_{t+1})\left\{\frac{\tau(k_{t+1})\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{1+\beta}\right)}{1 + \tau(k_{t+1})\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{1+\beta}\right)}\right\}. \quad (4.4)$$

In order to simplify the notation, I will define the function $f(\cdot)$ as follows

$$f(\tau(k_{t+1})) \equiv \left\{\frac{\tau(k_{t+1})\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{1+\beta}\right)}{1 + \tau(k_{t+1})\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1}{1+\beta}\right)}\right\} \quad (4.5)$$

so that the replacement rate discounted by one period can be expressed simply in terms of the transfer policies $\tau(k_t)$ and $\tau(k_{t+1})$. In fact

$$\frac{b_{t+1}}{(1+r_{t+1})} = \beta(1 - \tau(k_t))f(\tau(k_{t+1})). \quad (4.6)$$

I am now able to rewrite both the consumption and saving functions as follows

$$c_{1t} = \left(\frac{1}{1+\beta}\right)(1 - \tau(k_t))[1 + \beta f(\tau(k_{t+1}))]w_t, \quad (4.7)$$

$$c_{2t} = \left(\frac{\beta}{1+\beta}\right)(1 - \tau(k_t))[1 + \beta f(\tau(k_{t+1}))](1 + r_{t+1})w_t, \quad (4.8)$$

$$S_t = \left(\frac{\beta}{1+\beta}\right)(1 - \tau(k_t))[1 - f(\tau(k_{t+1}))]w_t. \quad (4.9)$$

Rewriting the consumption and saving functions in this way is very convenient, since it allows to write down the objective function of the generational government simply as a function on the policy functions $\tau(k)$. That is, the maximization problem can be rewritten as follows:

$$\begin{aligned} & \max_{0 < \tau(\cdot) < 1} u(c_{1t}, c_{2t}) \\ & = \lg \left\{ \left(\frac{1}{1+\beta} \right) (1 - \tau(k_t)) [1 + \beta f(\tau(k_{t+1}))] w_t \right\} \\ & \quad + \beta \lg \left\{ \left(\frac{\beta}{1+\beta} \right) (1 - \tau(k_t)) [1 + \beta f(\tau(k_{t+1}))] (1 + r_{t+1}) w_t \right\}, \end{aligned} \quad (4.10)$$

such that

$$(1+n)k_{t+1} = \left(\frac{\beta}{1+\beta} \right) (1 - \tau(k_t)) [1 - f(\tau(k_{t+1}))] w_t.$$

The first result can now be stated:

Proposition 1. *There exist a continuum of differentiable interior stationary policy function of the form*

$$\tau(k) = \left(\frac{\alpha}{1-\alpha} \right) [Ck^{-(1+\beta\alpha)/(1+\beta)} - 1],$$

where $C \geq 0$ is a constant (of integration) that can assume arbitrary (non-negative) values.

Proof. See Appendix A. \square

C is a constant of integration which can assume arbitrary non-negative values. Therefore, the equilibrium policy function is in fact a family of functions. That is, there are multiple stationary policies indexed by the free parameter C . Since C is non-negative, the payroll tax will be non-increasing in k .

The intuition behind the fact that the policy function is not increasing in the capital stock is the following: were the expected transfer increasing in k , current young would have an additional incentive to save in order to provide the next generation with a higher level of capital and therefore receive a higher old age transfer. But this cannot be an equilibrium, since the higher transfer in old age reduces the level of saving that young workers are willing to make.

In order for the equilibrium to be sustainable, the strategy has to induce a stable dynamic for the capital stock. Substituting the found policy function into the dynamic equation for the capital, we find the following relationship:

$$\underbrace{\beta k_{t+1} + C k_{t+1}^{1-\vartheta}}_{g(k_{t+1})} = \underbrace{(\beta/(1+n)) [k_t^\alpha - \alpha C k_t^{\alpha-\vartheta}]}_{h(k_t)} \quad (4.11)$$

where $\vartheta = (1 + \beta\alpha)/(1 + \beta)$. As (4.11) shows, the dynamic equation cannot be written in an explicit form. I therefore defined a function of k_{t+1} , $g(k_{t+1})$, and one of k_t , $h(k_t)$. The steady state capital stock will be that level of k that solves $g(k) = h(k)$.

The capital dynamic described by Eq. (4.11) is valid as long as the policy remains an interior solution of the problem, that is as long as $0 < \tau(k) < 1$. Therefore, in equilibrium the dynamic of the capital stock must be stable *and* such that the capital stock remains inside a given interval: the constraint $0 < \tau(k) < 1$ is satisfied as long as $k \in [\underline{k} = (\alpha C)^{1/\vartheta}, \bar{k} = C^{1/\vartheta}]$. Consider the case in which the capital stock reaches a level below \underline{k} . The strategy of Proposition 1 would imply $\tau > 1$, which is not feasible; therefore the generation has to play constrained (either $\tau = 0$ or $\tau = 1$). In this case, the solution would imply setting $\tau = 0$, since the utility with $\tau = 1$ is equal to minus infinity. By backward induction, this implies that all generations will pick $\tau = 0$. A similar argument runs when k hits the \bar{k} threshold. Therefore, unless the capital stock moves inside the interval $[\underline{k}, \bar{k}]$, the solution given by the strategy of Proposition 1 is not an equilibrium and the PAYGO is not sustainable through a Markov equilibrium.

It is easy to check from (4.11) that for a wide range of parameter values there is no steady state. When a steady state exists, there are usually two of them: one for low values of k and unstable, the other one for higher values of k and stable. Figure 1 shows one stable steady state. The figure assumes $n = 0.0$, $\alpha = 0.25$, $\beta = 0.9$ and $C = 0.42$. On the horizontal axis the value of k is reported. Where the $g(k) - h(k)$ function crosses the zero value, we have the stable steady state (the unstable one is not shown, but it would lie on the left of the stable one). In this case, the steady state capital stock is approximately 0.2, the equilibrium payroll tax 4%, and $[\underline{k}, \bar{k}] = [0.03, 0.26]$. Moreover, it is easy to show that the capital converges monotonically to the equilibrium (there are no cycles). The value of the steady state capital stock, 0.2, compares with the competitive equilibrium level equal in this example to 0.252.

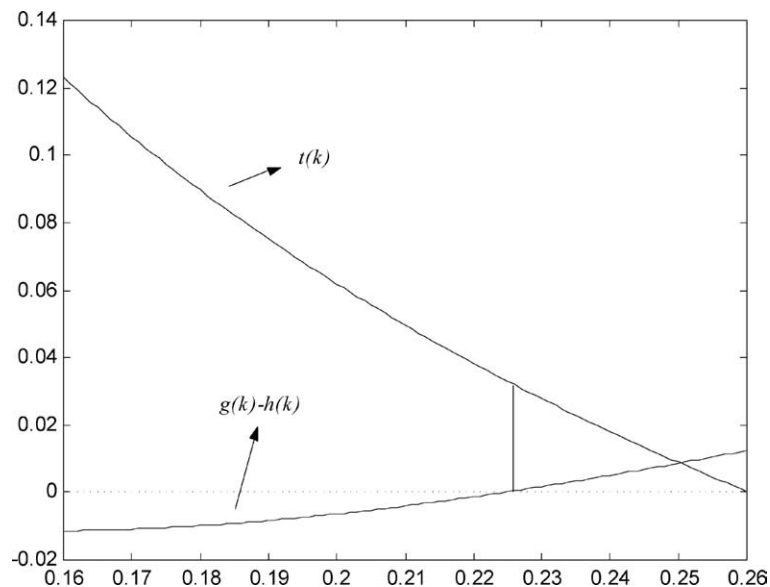


Fig. 1. Stable steady state ($\alpha = 0.25$, $\beta = 0.9$, $C = 0.42$).

The dynamic characteristics of the equilibrium are similar for combinations of the parameters close to the ones used in Fig. 1. In particular, if agents have a higher β , they are willing to save more and thus they accept a lower equilibrium transfer. An increase in n reduces per capita saving and thus steady state capital stock; this implies a higher level of steady state transfer (note that n has no direct effect on the equilibrium transfer function). On the other hand, an increase in C increases the level of the equilibrium transfer and hence reduces the steady state capital stock. An increase in the capital share, α , reduces savings and capital accumulation (because the wage bill is lower), but it reduces also the wage bill out of which the payroll tax is levied; the overall effect is to decrease the steady state level of capital and to increase the steady state payroll tax.

In this section, assuming a logarithmic utility function, I showed that there are cases in which the Markov solution implies positive transfers to the old both in the steady state and along the path to the steady state. However, a stable steady state exists for a limited set of parameters. For the combinations of the parameter values such that the model has no steady state, the capital stock will reduce (or grow) over time, while the transfer will increase (or decrease) until the tax hits the 100% ceiling (or the 0% floor). In these cases there is no equilibrium with positive transfer and there is no sustainable PAYGO.

In the model, even if the economy is dynamically efficient, young people may set a positive payroll tax for two reasons:

- (i) one reason is the “saving monopolist” one, that is the fact that the transfer of resources from the young to the old reduces the level of capital and—if the return per unit of capital has an elasticity greater than one—a reduction in saving increases the total return more than proportionally, thereby increasing lifetime income of the young generation;
- (ii) the second one is that a positive transfer to the elderly will reduce the level of capital and will induce in equilibrium an increase in future payroll taxes (this can be seen from the first order condition (3) reported in Appendix A). This second element is missing in the existing literature.⁶

It should be stressed that these results are obtained under the assumptions of symmetric equilibrium and of continuous and differentiable policy function. The assumption that the equilibrium is symmetric is reasonable, given the stationarity of the problem faced by each generation; it is however a constraint. In general, non-stationary equilibria may exist and may present more complicate dynamics. Similarly, the assumption that the policy function is continuous and differentiable imposes additional restrictions on the equilibrium strategy, since it implies for example that the payroll tax cannot jump.

5. A more general case

In this section I analyze the model under the assumption that agents have a more general utility function. In particular, I will assume an isoelastic utility function. With the isoelas-

⁶ For example, Boldrin and Rustichini (2000) assume agents either decide to pay a given payroll tax (set by the previous generation) or not to pay at all.

tic utility, the saving function depends on the interest rate and the problem complicates substantially. In particular, I could not solve it analytically and I characterized the solution through numerical simulation.

Through numerical simulation I am able to characterize the solution for different values of the parameters and assess their effect on the equilibrium outcome. The results, however, must be taken with great caution. In particular, the simulations are run under the assumption that an equilibrium does in fact exist. While this is a rather strong assumption, the simulations generally converge to a unique equilibrium. This suggests that the dynamic of the model with logarithmic utility function may be specific to the log formulation and to the linearity that it introduces into the problem.

5.1. The approach

I continue to assume a Cobb–Douglas production function. The utility function I consider now is a CRRA as follows:

$$U(c_{1t}, c_{2t+1}) = \left(\frac{1}{1-\sigma} \right) \{c_{1t}^{(1-\sigma)} + \beta c_{2t+1}^{(1-\sigma)}\} \quad (5.1)$$

where $\sigma > 0$. For $\sigma = 1$ the CRRA utility function is equal to the logarithmic one.

The simulation program uses polynomials to approximate two functions:

- (1) the first one is the equilibrium Markov strategy as a function of the capital stock, $\tau(k)$. The constraint that the equilibrium is symmetric is imposed by assuming $\tau_t(\cdot) = \tau_{t+1}(\cdot) = \tau(\cdot)$; the constraint that $0 < \tau(k) < 1$ is imposed combining the polynomial with a logistic function;⁷
- (2) the second function to approximate is the explicit version of the capital accumulation condition $k_{t+1} = \Psi(k_t)$. $\Psi(k_t)$ is the (possibly) explicit form of the capital accumulation equation $(1+n)k_{t+1} = S_t(k_t, k_{t+1}, \tau(k_t), \tau(k_{t+1}))$. The constraint $k_{t+1} \geq 0$ is imposed through exponentiation.

The program uses Chebyshev Polynomials to approximate the two functions, $\tau(k)$ and $\Psi(k)$, and obtain $\hat{\tau}(k)$ and $\hat{\Psi}(k)$. Chebyshev Polynomials have—on top of the well known asymptotic properties of polynomial approximations—the advantage of evaluating the functions at a given limited number (equal to the order of the polynomial) of points such that the approximated functions converge to the true functions as more points are used.

The Lagrangian of the maximization problem can then be written as follows:

$$L(k_t, \hat{k}_{t+1}, \hat{\tau}(k_t), \hat{\tau}(\hat{k}_{t+1})) = \left(\frac{1}{1-\sigma} \right) \{ \hat{c}_{1t}^{(1-\sigma)} + \beta \hat{c}_{2t+1}^{(1-\sigma)} \} + \lambda [(1+n) \cdot \hat{k}_{t+1} - \hat{S}_t] \quad (5.2)$$

⁷ It is worth reminding that the operator T defined in (3.1) is not a contraction; this implies that it is not generally true that the equilibrium strategy will converge to a stationary function and I cannot use value function types of approaches to approximate it.

where

$$\begin{aligned}\hat{c}_{1t}(k_t, \hat{k}_{t+1}, \hat{\tau}(k_t), \hat{\tau}(\hat{k}_{t+1})) &= w_t(1 - \hat{\tau}(k_t)) - \hat{S}_t, \\ \hat{c}_{2t}(k_t, \hat{k}_{t+1}, \hat{\tau}(k_t), \hat{\tau}(\hat{k}_{t+1})) &= \hat{S}_t(1 + \hat{r}_{t+1}) + \hat{b}_{t+1}, \\ \hat{S}_t(k_t, \hat{k}_{t+1}, \hat{\tau}(k_t), \hat{\tau}(\hat{k}_{t+1})) &= \frac{\{\beta(1 + \hat{r}_{t+1})\}^{1/\sigma} w_t(1 - \hat{\tau}(k_t)) - \hat{b}_{t+1}}{(1 + \hat{r}_{t+1}) + \{\beta(1 + \hat{r}_{t+1})\}^{1/\sigma}}, \\ \hat{b}_{t+1}(\hat{k}_{t+1}, \hat{\tau}(\hat{k}_{t+1})) &= \hat{\tau}(\hat{k}_{t+1}) \hat{w}_{t+1}(1 + n),\end{aligned}$$

where $\hat{k}_{t+1} = \hat{\Psi}(k_t)$, $\hat{r}_{t+1} = r(\hat{k}_{t+1})$ and $\hat{w}_{t+1} = w(\hat{k}_{t+1})$. To solve the model, I calculate the two first order conditions $dL/d\tau_t = 0$ and $dL/dk_{t+1} = 0$, evaluate them at the Chebyshev interpolation nodes and compute the sum of squared residuals (given an initial guess for the parameters of the polynomials). Finally, I use a non-linear estimation procedure, usually used in maximum likelihood estimations, to minimize the sum of squared residual through the choice of the parameters of the polynomials.⁸ I use 10th order polynomials to approximate both the policy function and the capital accumulation function. I obtain similar results when using higher order polynomials.

5.2. Results

In order to check the quality of the program I first simulated a competitive standard two period OLG model with and without social security. I used the technique outlined above to approximate the capital accumulation function and I obtained the smooth function that we know. The steady state value of the capital stock is the correct one (in fact, it can be computed analytically). Unfortunately, in order to check the correctness of the program, I cannot replicate the logarithmic utility function case for which I have the close form solution. This strategy is precluded by the fact that the close form solution is not unique (there is in fact a continuum of policy functions indexed by the value of the constant of integration C).

In general, for realistic values of the parameters, that is for values of $\alpha \in (0, 1)$, $\beta \in (0, 1]$ and $\sigma \geq 0.5$, the program converges—for almost any initial guess—to the same solution characterized by a unique and stable steady state. When it does not converge to a unique steady state, the program generally does not find any steady state.

Figures 2 and 3 show the capital accumulation and the policy functions for two different values of σ ($\sigma = 1.5$ in Fig. 2 and $\sigma = 0.5$ in Fig. 3), while the other parameters are: $\alpha = 0.25$, $\beta = 0.7$, $n = 0.0$. The plots show that the capital accumulation is significantly flattened out with respect to the competitive case. Moreover, while for $\sigma < 1$ the capital accumulation function is almost flat, for $\sigma > 1$ it is slightly decreasing. In this second case, the convergence to the equilibrium comes with cycles.

The effects of changing the parameters values are very similar to the logarithmic utility case, regardless of the fact that σ is greater or smaller than 1. In fact, as in the logarithmic

⁸ The program is written in Matlab, and the maximization tool used is the command `leastsq`. For 10th order polynomials, I consider the problem as approximately solved when the sum of squared residual is below 10^{-5} . The program is available from the author upon request.

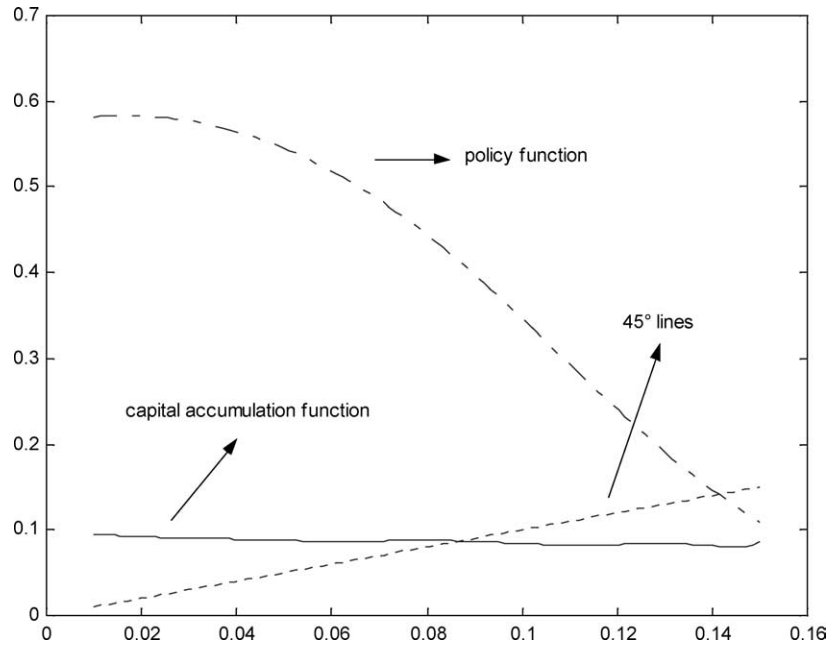


Fig. 2. Capital accumulation and policy function ($\sigma = 1.5, \alpha = 0.25, \beta = 0.7$).

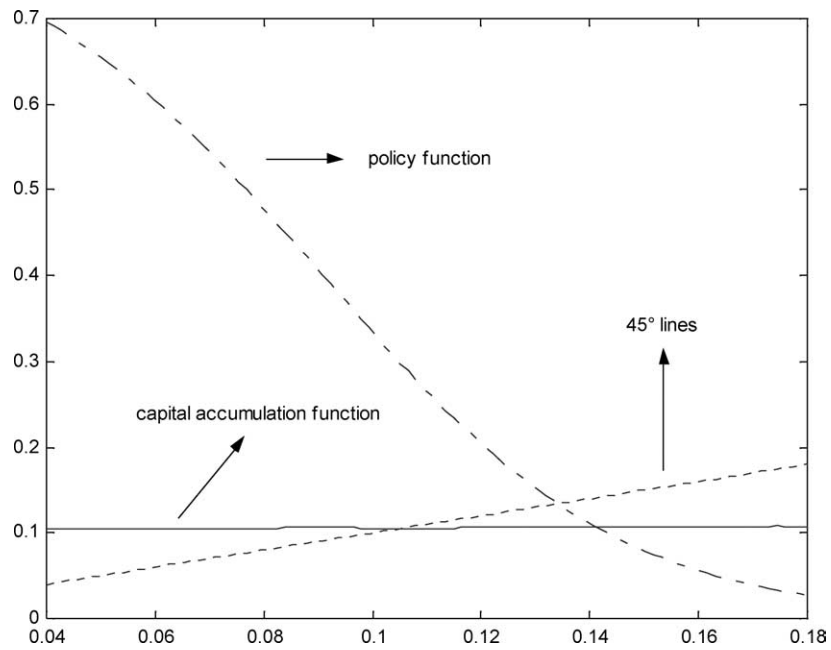


Fig. 3. Capital accumulation and policy function ($\sigma = 0.5, \alpha = 0.25, \beta = 0.7$).

utility case, an increase in α (the capital share) on one hand reduces savings and capital accumulation (because the wage bill is lower), on the other it reduces the wage bill out of which the payroll tax is levied: the overall effect is to increase the payroll tax and to reduce the steady state level of capital. An increase in n reduces per capita saving and thus steady state capital stock; this implies a higher level of steady state transfer. Finally, the effect of a change in β is to increase the saving level and the equilibrium steady state capital stock.

In general, the equilibrium steady state level of capital will be below the one of the competitive OLG model (the standard Diamond–Samuelson). In fact, Fig. 2 (with $\sigma = 1.5$) steady state capital level is equal to 0.085 (with a steady state payroll tax level equal to 0.428), while the corresponding steady state level of the competitive OLG model (with payroll tax equal to zero) is approximately 0.25. With $\sigma = 0.5$, the corresponding numbers are 0.101 and 0.15 (with a steady state payroll tax level equal to 0.295).

Assuming the existence of a social security scheme in the competitive OLG model with a payroll tax equal to 0.428 for $\sigma = 1.5$ and 0.295 for $\sigma = 0.5$, the steady state capital stock ends up being lower than the one in the political game (0.04 for $\sigma = 1.5$ and 0.08 for $\sigma = 0.5$). This result, which looks somehow surprising, reflects the presence of the punishment (in the form of reduced old age pension benefits) related to the accumulation of capital. With respect to the competitive model, where the payroll tax is exogenously given, in the political game the effect on old age consumption of a change in the level of savings is partly balanced out by the induced change in old age benefits. Therefore in equilibrium a higher level of saving is necessary to achieve consumption smoothing.

6. Conclusions

This paper has considered a standard overlapping generations model where individuals live and consume for two periods. They work during the first period and are retired in the second one. In this context I tried to understand why selfish young generations may be willing to pay a retirement income to their parents by means of a pay-as-you-go (PAYGO) social security system. In fact, unless the economy without the PAYGO scheme is on a dynamically inefficient path, young workers should be better off discontinuing the PAYGO scheme and investing the payroll tax in productive capital. Even when the economy without the PAYGO scheme is on a dynamically inefficient path, young workers should be better off discontinuing the PAYGO scheme and setting up a new one.

A number of authors have shown that simple intergenerational games in which in each period the current young generation plays as a dictator are able to deliver political equilibria with social security even if the underlying competitive equilibrium is dynamically efficient. This paper extends these results to equilibria that are Markov (i.e. they depend only upon the period t state variable, which is the stock of capital). The intuition is similar to the one developed in previous contributions. The equilibrium strategy, which implies a reduction in payroll taxes as the capital stock increases, is simply the implementation of a punishment strategy in a Markovian context. It is worth noting that this strategy actually leads to a higher steady state level of capital than in the model where taxes are exogenously given: in equilibrium, since pension benefits are a decreasing function of the capital stock, a higher

level of saving than in the competitive model is necessary in order to achieve consumption smoothing.

Acknowledgments

I wish to thank Christophe Chamley, Vincenzo Galasso, Simon Gilchrist, Kevin Lang, Larry Kotlikoff and Debraj Ray for comments and discussions at different stages of this work, and participants at seminars in Boston, Copenhagen, Leuven and at the SED 2000 conference. I owe special thanks to one editor of this journal. The views expressed in this paper are my own and should not be held to represent those of the Bank of Italy. I am responsible for any error.

Appendix A

Proof of Proposition 1

In order to proof the proposition, I look for an interior solution such that $0 < \tau(\cdot) < 1$. The proof will be performed in *two steps*. In the *first step* I will compute the first order condition using as a constraint the first derivative with respect to τ of the logarithm of the capital accumulation equation (instead of using the constraint in levels). In this way I am able to obtain a differential equation for the policy variable $\tau(k)$ which can be solved explicitly. In the *second step* I will show that the specific solution found satisfies the first order necessary and second-order sufficient conditions of the problem and therefore it is a proper solution.

First step. Using the fact that $w_t = (\frac{1-\alpha}{\alpha})(1+r_t)k_t$ and the properties of the logarithmic function, the problem in (4.10) can be restated as follows:

$$\begin{aligned} \max_{0 < \tau_t < 1} u(c_{1t}, c_{2t}) \\ = A + (1 + \beta) \{ \lg(1 - \tau_t) + \lg[1 + \beta f(\tau_{t+1})] \} + \beta \lg(1 + r_{t+1}) \end{aligned} \quad (\text{A.1})$$

such that

$$k_{t+1} = \left(\frac{1}{1+n} \right) \left(\frac{\beta}{1+\beta} \right) (1 - \tau_t) [1 - f(\tau_{t+1})] w_t,$$

where

$$A = \left\{ \lg \left(\frac{1}{1+\beta} \frac{1-\alpha}{\alpha} \right) + \beta \lg \left(\frac{\beta}{1+\beta} \frac{1-\alpha}{\alpha} \right) + (1+\beta) [\lg k_t + \lg(1+r_t)] \right\}.$$

Differentiating Eq. (A.1) with respect to τ_t we get

$$\left[(1 + \beta) \frac{\beta f'_{\tau_{t+1}}}{1 + \beta f(\tau_{t+1})} \frac{d\tau_{t+1}}{dk_{t+1}} + \beta \frac{1}{1 + r_{t+1}} \frac{dr_{t+1}}{dk_{t+1}} \right] \frac{dk_{t+1}}{d\tau_t} = (1 + \beta) \frac{1}{1 - \tau_t}, \quad (\text{A.2})$$

and differentiating the logarithm of Eq. (A.2) with respect to τ_t and solving for $dk_{t+1}/d\tau_t$ we have

$$\frac{dk_{t+1}}{d\tau_t} = -\frac{1}{(1-\tau_t)}k_{t+1} \left/ \left(1 + k_{t+1} \frac{f'_{\tau_{t+1}}}{1-f(\tau_{t+1})} \frac{d\tau_{t+1}}{dk_{t+1}} \right) \right. \quad (\text{A.3})$$

Substituting Eq. (A.3) into (A.2), and using the fact that

$$(1+r_{t+1}) = \alpha k_{t+1}^{\alpha-1} \implies \beta \frac{1}{1+r_{t+1}} \frac{dr_{t+1}}{dk_{t+1}} = -\frac{\beta(1-\alpha)}{k_{t+1}},$$

we can rewrite (A.2) to get

$$(1+\beta) \left[\frac{\beta f'_{\tau_{t+1}}}{1+\beta f(\tau_{t+1})} + \frac{f'_{\tau_{t+1}}}{1-f(\tau_{t+1})} \right] \frac{d\tau_{t+1}}{dk_{t+1}} = -(1+\beta\alpha) \frac{1}{k_{t+1}}. \quad (\text{A.4})$$

Integrating both sides of (A.4) we get

$$\lg \left[\frac{1+\beta f(\tau(k_{t+1}))}{1-f(\tau(k_{t+1}))} \right] = -\left(\frac{1+\beta\alpha}{1+\beta} \right) \lg k_{t+1} + C \quad (\text{A.5})$$

where C is a non-negative constant of integration.

Recalling the definition of $f(\tau(k_{t+1}))$ from Eq. (4.5), Eq. (A.5) can be rewritten in the following way:

$$\left[1 + \left(\frac{1-\alpha}{\alpha} \right) \tau(k_{t+1}) \right] = C k_{t+1}^{-(1+\beta\alpha)/(1+\beta)}; \quad (\text{A.6})$$

then, solving for $\tau(k_{t+1})$, we obtain the strategy reported in Proposition 1.

Second step. I now restate the problem in the Lagrangian form as follows:

$$L = A + (1+\beta) \left\{ \lg(1-\tau_t) + \lg[1+\beta f(\tau_{t+1})] \right\} + \beta \lg(1+r_{t+1}) + \\ -\lambda \left\{ k_{t+1} - \left(\frac{1}{1+n} \right) \left(\frac{\beta}{1+\beta} \right) (1-\tau_t) [1-f(\tau_{t+1})] w_t \right\}$$

where λ is the Lagrange multiplier. Assuming that

$$\tau(k) = \left(\frac{\alpha}{1-\alpha} \right) \left[C k^{-(1+\beta\alpha)/(1+\beta)} - 1 \right],$$

it is easy to show that the following system (representing the set of necessary conditions) is always satisfied:

$$\frac{dL}{d\tau_t} = 0, \quad \frac{dL}{dk_{t+1}} = 0, \\ k_{t+1} = \left(\frac{1}{1+n} \right) \left(\frac{\beta}{1+\beta} \right) (1-\tau(k_t)) [1-f(\tau(k_{t+1}))] w_t.$$

The proposed solution satisfies also the second-order sufficient conditions. In fact, it is easy to show that $Z_{\tau k} = Z_{k\tau} \equiv d^2L/(d\tau_t dk_{t+1}) = d^2L/(dk_{t+1} d\tau_t) = 0$, so that the bordered Hessian of the Lagrangian is equal to:

$$-g_{\tau}(g_{\tau}Z_{kk} - g_k Z_{\tau k}) + g_k(g_{\tau}Z_{\tau k} - g_k Z_{\tau\tau}) = -g_{\tau}^2 Z_{kk} - g_k^2 Z_{\tau\tau}$$

where $Z_{kk} \equiv d^2L/dk_{t+1}^2$, $Z_{\tau\tau} \equiv d^2L/d\tau_t^2$; g_τ and g_k are the derivatives of the constraint with respect to τ_t and k_{t+1} respectively. It is easy to show that $-g_\tau^2 Z_{kk} - g_k^2 Z_{\tau\tau} > 0$ on the relevant range $[\underline{k}, \bar{k}]$. Therefore the bordered Hessian is negative definite, that is the second-order sufficient condition for a maximum is satisfied.

References

- Azariadis, C., Galasso, V., 1997. Fiscal constitutions and the determinants of intergenerational transfers. Working paper 97-71. Universidad Carlos III de Madrid.
- Boldrin, M., Rustichini, A., 2000. Political equilibria with social security. *Review of Economic Dynamics* 3.
- Coolley, T., Soares, J., 1998. A positive theory of social security based on reputation. *Journal of Political Economy* 107.
- Galasso, V., 1998. The US social security system: What does political sustainability imply? *Review of Economic Dynamics*. In press.
- Grossman, G., Helpman, E., 1998. Intergenerational redistribution with short lived governments. *Economic Journal* 108.
- Krusell, P., Quadrini, V., Rios-Rull, V., 1997. Politico-economic equilibrium and economic growth. *Journal of Economic Dynamics and Control* 21.